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Weak Disorder Expansion of the Invariant Measure for the One-Dimensional Anderson Model

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We show that the formal perturbation expansion of the invariant measure for the Anderson model in one dimension has singularities at all energies $E_0 = 2\cos\pi(p/q)$; we derive a modified expansion near these energies that we show to have finite coefficients to all orders. Moreover, we show that the first $q-3$ of them coincide with those of the "naive" expansion, while there is an anomaly in the $(q-2)$th term. This also gives a weak disorder expansion for the Liapunov exponent and for the density of states. This generalizes previous results of Kappus and Wegner and of Derrida and Gardner.

KEY WORDS: Anderson model; invariant measure; Liapunov exponent; density of states; weak disorder expansion.

1. INTRODUCTION

Over the last few years the mathematical understanding of the properties of one of the oldest models for electron transport in disordered crystals, the Anderson model,[1] has greatly advanced. In particular, there are detailed results on the nature of the density of states available. However, essentially all results concern the localized regime, i.e., the domain of parameters where the disorder effectively dominates the behavior of the system. That is, they regard the model in one dimension[2] or they are confined to large disorder or high energy.[3] Virtually nothing is known when the disorder is "weak."

In this paper we take up some earlier attempts[4-6] to derive a perturbation expansion for the density of states in one dimension for weak disorder.

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To be specific, we study the Hamiltonian

$$H = H_0 + \lambda V \tag{1.1}$$

on $l^2(\mathbf{Z})$ where

$$(H_0 u)(n) = u(n+1) + u(n-1) \tag{1.2}$$

and V is a diagonal matrix with diagonal elements $V(n)$, $n \in \mathbf{Z}$, that are independent, identically distributed random variables with a common probability distribution μ. We will choose for definiteness μ such that

$$\int d\mu(V)\, V = 0, \qquad \int d\mu(V)\, V^2 = 1 \tag{1.3}$$

Furthermore, we will assume that μ has finite moments of all orders.

The eigenvalue equation associated with H is

$$u(n+1) + u(n-1) + \lambda V(n)\, u(n) = Eu(n) \tag{1.4}$$

Introducing $Z(n) \equiv u(n)/u(n-1) \in \dot{\mathbf{R}}$, we may write this as a recursion relation for $Z(n)$:

$$Z(n+1) = E - \lambda V(n) - 1/Z(n) \tag{1.5}$$

The Liapunov exponent $\gamma(E)$ and the density of states $N(E)$ are related to the large-n behavior of $Z(n)$. If we define

$$\tilde{\gamma}(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln Z(i) \tag{1.6}$$

then

$$\gamma(E) = \operatorname{Re} \tilde{\gamma}(E) \tag{1.7}$$

$$N(E) = \pi \operatorname{Im} \tilde{\gamma}(E) \tag{1.8}$$

Equation (1.5) defines a Markov process and we may expect to compute large-n limits from an invariant measure associated with (1.5). Indeed, Furstenberg's theorem[7] asserts that, for $\lambda \neq 0$, there is a unique invariant measure $d\nu_{\lambda,E}(x)$ on $\dot{\mathbf{R}}$, i.e., a measure satisfying

$$\int_{\dot{\mathbf{R}}} d\nu_{\lambda,E}(x)\, f(x) = \mathbf{E} \int_{\dot{\mathbf{R}}} d\nu_{\lambda,E}(x)\, f\left(E - \lambda V - \frac{1}{x}\right) \tag{1.9}$$

for all bounded, measurable functions f. Here $\mathbf{E}$ denotes the expectation with respect to $d\mu(V)$. The invariant measure is actually continuous and hence supported by $\mathbf{R}$. Moreover,

$$\tilde{\gamma}(E) = \int_{\mathbf{R}} \ln x \, d\nu_{\lambda,E}(x) \tag{1.10}$$

This invariant measure is thus a basic object to compute, and we will concentrate on deriving the perturbation expansion for $d\nu_{\lambda,E}(x)$, more precisely for the density of this measure.

Problems with a straightforward perturbation expansion in λ as proposed by Thouless[4] were first discovered by Kappus and Wegner,[5] who noticed that the leading coefficient in λ was inadequate in the center of the band ($E=0$) and that the differentiated density of states $n_\lambda(E)$ exhibited a discontinuity there. They called this phenomenon an anomaly. Derrida and Gardner,[6] looking at the invariant measure, extended this result. They found that at energies $E = \pm 1$, the next-to-leading coefficient of the Thouless expansion was incorrect, and does exhibit a discontinuity there. They also conjectured that such anomalies should indeed occur at all energies of the form $E = 2\cos \pi(p/q)$, with p, q relatively prime integers.

In the present paper we give a rather complete analysis of this situation, at least at the level of formal perturbation theory. First, in Section 2 we derive the equations for the density of the invariant measure and explain why problems should be anticipated at the special energies mentioned above. We also indicate how to resolve these problems. Then, in Section 3, we derive the equations governing the coefficients of the formal perturbation expansion and show that, for energies $E = 2\cos\pi\alpha$ with α irrational, they have a unique solution with finite coefficients to all orders. However, the coefficient of order $n = q$ will be seen to diverge as $\alpha \to p/q$, with p and q relatively prime integers. Furthermore, we will see that for α rational the equations no longer determine a unique solution. In Section 3 we derive the equations for the modified expansion for energies near $E_0 = 2\cos\pi(p/q)$. We show that those admit a unique solution with finite coefficients to all orders. Moreover, we show that they differ from the naive ones only at order $n \geqslant q-2$.

While this provides a detailed understanding of the properties of the formal perturbation expansion, we have not been able to prove that this expansion is *asymptotic* to the true invariant measure, even if we were to impose further restrictions on the distribution μ. Such a result would be very desirable. We discuss our partial results at the end of Section 4.

2. EQUATIONS FOR THE INVARIANT MEASURE

A solution of (1.9) exists for all $\lambda > 0$ as a probability measure. Under weak conditions on the distribution μ (decay of the characteristic function), this measure even has a density.[2] If we assume the existence of a density $\varphi_{\lambda,E}(x)$, (1.9) implies that it must satisfy the equation

$$\varphi_{\lambda,E}(x) = \mathbf{E}\left[\frac{1}{(E-\lambda V-x)^2}\,\varphi_{\lambda,E}\left(\frac{1}{E-\lambda V-x}\right)\right] \tag{2.1}$$

We find it convenient to write this equation as

$$\varphi_{\lambda,E}(x) = B_{\lambda,E_0,\varepsilon}\,\varphi_{\lambda,E}(x) \tag{2.2}$$

where

$$B_{\lambda,E_0,\varepsilon} = \mathbf{E}\left[\exp\left(V\lambda\frac{d}{dx} - \varepsilon\lambda^2\frac{d}{dx}\right)\right] T_{E_0} \tag{2.3}$$

$$(T_{E_0}f) = \frac{1}{(E_0-x)^2}\, f\left(\frac{1}{E_0-x}\right) \tag{2.4}$$

and $E = E_0 + \lambda^2\varepsilon$, with E_0 chosen at our convenience. We will see that the properties of T_{E_0} are essentially responsible for the appearance of "anomalies." Defining $\tau_{E_0}(x) = 1/(E_0 - x)$, we have

$$(T_{E_0}f)(x) = f(\tau_{E_0}(x))\, d\tau_{E_0}(x)/dx$$

The map τ_{E_0} for $-2 < E_0 < 2$ has no fixed points, and depending on E_0, it is either cyclic or ergodic. These facts are most easily understood by mapping $\dot{\mathbf{R}}$ to the circle. We put $E_0 = 2\cos\pi\alpha$, and, following refs. 5 and 6, change variables from $x \in \mathbf{R}$ to $\theta \in S_1$, where by S_1 we mean the circle of circumference π, via

$$x = \sin(\theta + \pi\alpha)/\sin\theta \tag{2.5}$$

Thus we define a map J_α by

$$(J_\alpha f)(\theta) = f(x)\, dx/d\theta \tag{2.6}$$

Then T_{E_0} is mapped to

$$\tau_\alpha = J_\alpha T_{E_0} J_\alpha^{-1}$$

and

$$(\tau_\alpha g)(\theta) = g(\theta - \pi\alpha) \tag{2.7}$$

For $\alpha \notin \mathbf{Q}$, τ_α is thus an ergodic map on the circle and Lebesgue measure is the only mesure invariant under it. On the other hand, if $\alpha = p/q$, τ_α is cyclic with period q and there are infinitely many invariant measures for it. These statements also translate to spectral properties of T_{E_0} (here we think of T_{E_0} as an operator on $L^1(\mathbf{R}, dx)$): if $E_0 = 2\cos \pi\alpha$ with α irrational, the spectrum of T_{E_0} is the unit circle and one is its only eigenvalue, which has multiplicity one. If $\alpha = p/q$, the spectrum of T_{E_0} consists of the qth roots of unity; all these eigenvalues are infinitely degenerate.

In particular, the equation $T_{E_0} f = f$ always has the (normalized) solution

$$\tilde{\varphi}_{E_0}^{(0)}(x) = \frac{(4 - E_0^2)^{1/2}}{2\pi} \frac{1}{x^2 - E_0 x + 1} \tag{2.8}$$

If α is irrational, this solution is unique; otherwise, there exist infinitely many others. This indicates that at the special energies $E_0 = 2\cos \pi(p/q)$ we should anticipate problems if we try to perturb around T_{E_0}. This will be made explicit in the next section.

Let us now indicate how we may overcome this difficulty. The crucial idea is to iterate Eq. (2.2) q times if $E_0 = 2\cos \pi(p/q)$. This gives

$$\varphi_{\lambda,E}(x) = B^q_{\lambda,E_0,\varepsilon} \varphi_{\lambda,E}(x) \tag{2.9}$$

As $\lambda \to 0$, $B^q_\lambda \to I$. More importantly,

$$A_{\lambda,E_0,\varepsilon} \equiv (B^q_{\lambda,E_0,\varepsilon} - I)/\lambda^2 \tag{2.10}$$

converges (strongly on a dense set) to a differential operator with zero as a simple eigenvalue! (This fact will be established in Section 4.) Thus, we may consider $A_{\lambda,E_0,\varepsilon}$ as a small perturbation of $A_{0,E_0,\varepsilon}$, and hope to derive a viable perturbation expansion starting from the equation

$$A_{\lambda,E_0,\varepsilon} \varphi_{\lambda,E}(x) = 0 \tag{2.11}$$

Again we will show this to be the case, at the level of formal perturbation theory, in Section 4.

3. THE "NAIVE" PERTURBATION THEORY

The obvious attempt to find a weak-disorder expansion for $\varphi_{\lambda,E}(x)$ is to (formally) write

$$\varphi_{\lambda,E}(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \varphi_E^{(n)}(x) \tag{3.1}$$

where $\varphi_{\lambda,E}$ is normalized by requiring its integral to be one, and to plug this ansatz into Eq. (2.2). A simple application of the Leibnitz rule gives then

$$\varphi_E^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\partial^k}{\partial \lambda^k} B_{\lambda,E} \right)_{\lambda=0} \varphi_E^{(n-k)}(x) \tag{3.2}$$

or, with

$$\left(\frac{\partial^k}{\partial \lambda^k} B_{\lambda,E} \right)_{\lambda=0} = \mathbf{E}(V^K) \frac{d^k}{dx^k} T_E \tag{3.3}$$

$$(I - T_E)\, \varphi_E^{(n)}(x) = \sum_{k=2}^{n} \binom{n}{k} \mathbf{E}(V^k) \frac{d^k}{dx^k} T_E \varphi_E^{(n-k)}(x) \tag{3.4}$$

It is useful to map these equations to the corresponding ones on the circle, i.e., denoting

$$h_{\lambda,E}(\theta) = (J_\alpha \varphi_{\lambda,E})(\theta) \tag{3.5}$$

(3.4) becomes

$$(1 - \tau_\alpha)\, h_E^{(n)}(\theta) = \sum_{k=2}^{n} \mathbf{E}(V^k) \binom{n}{k} \left(\frac{(\partial/\partial\theta) \sin^2 \theta}{\sin \pi\alpha} \right)^k \tau_\alpha h_E^{(n-k)}(\theta) \tag{3.6}$$

Introducing the Fourier coefficients of h,

$$\hat{h}_E^{(n)}(m) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} d\theta\, e^{-2im\theta} h_E^{(n)}(\theta)$$

we have that (3.6) implies that

$$(1 - e^{-2\pi i m\alpha})\, \hat{h}_E^{(n)}(m) = \sum_{k=2}^{n} \binom{n}{k} \frac{\mathbf{E}(V^k)}{(\sin \pi\alpha)^k} \sum_{l=-\infty}^{\infty} [D^k]_{ml}\, \hat{h}_E^{(n-k)}(l)\, e^{2\pi i \alpha l} \tag{3.7}$$

where

$$D_{ml} = \tfrac{1}{4} im(\delta_{m,l+1} + \delta_{m,l-1} - 2\delta_{l,m}) \tag{3.8}$$

If α is irrational, $1 - e^{-2\pi i m\alpha} \neq 0$ for all $m \neq 0$, and the system of equations (3.7) has a unique solution with the normalization $\hat{h}(0) = 1$. More precisely, we have the following result:

Lemma 3.1. If α is irrational, (3.7) has a unique normalized solution. Moreover,

(i) $\hat{h}_E^{(0)}(m) = \delta_{m,0}$

(ii) $\hat{h}_E^{(n)}(0) = \delta_{n,0}$

(iii) $\hat{h}_E^{(n)}(m) = 0$ if $|m| > n$

Proof. Since α is irrational, $1-e^{-2\pi i\alpha m}$ is zero only for $m=0$. But for $m=0$, the right-hand side of (3.7) equals

$$\sum_{k=2}^{n}\binom{n}{k}\frac{\mathbf{E}V^k}{(\sin\pi\alpha)^k}\int_{-\pi/2}^{\pi/2}\left(\frac{d}{d\theta}\sin^2\theta\right)^k\tau_\alpha h_E^{(n-k)}(\theta)$$

which is zero whenever $h_E^{(n-k)}(\theta)$ are periodic functions, which is the case by construction. Thus, a finite solution exists. Only the $\hat{h}_E^{(n)}(0)$ are not uniquely determined by the equations. But since the normalization of $h_{E,\lambda}(\theta)$ implies

$$\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\hat{h}_E^{(n)}(0)=1 \qquad \text{for all } \lambda$$

we are led to choose $\hat{h}_E^{(0)}(0)=1$ and $\hat{h}_E^{(n)}(0)=0$ for $n>0$. This gives (i) and (ii). Property (iii) follows from (i) using induction and the structure of the matrix D_{lm}.

If $\alpha=p/q$, Lemma 3.1 fails, since if m is a multiple of q, $1-e^{-2\pi im\alpha}=0$. Therefore, the recursion (3.7) no longer determines the solution uniquely. One might be tempted to extend the solutions $h_E^{(n)}(\theta)$ by continuity to these energies. Define, for $\alpha=p/q$, p and q relatively prime, and sequences $\alpha_i\to p/q$, $\alpha_i\notin\mathbf{Q}$, the limits

$$\lim_{i\to\infty}h_{E_i}^{(n)}(\theta)\equiv\tilde{h}_E^{(n)}(\theta) \tag{3.9}$$

Lemma 3.2. For $n<q$ the limits defined in (3.9) exist and are independent of the sequences α_i.

Proof. For any $\alpha_i\notin\mathbf{Q}$, $\hat{h}_{E_i}^{(n)}(m)=0$ if $|m|>n$. Thus, $\lim_{i\to\infty}\hat{h}_{E_i}^{(n)}(m)=0$ if $|m|>n$. On the other hand, if $m\leqslant n<q$,

$$|1-e^{-2\pi im\alpha}|\geqslant|1-e^{-2\pi i(p/q)m}|-|1-e^{-2\pi im(\alpha_i-p/q)}|\geqslant C>0$$

if i is large enough, so that, since

$$\hat{h}_{E_i}^{(n)}(m)=\frac{1}{1-e^{-2\pi i\alpha_i m}}\sum_{k=2}^{n}\binom{n}{k}\sum_l(D^k)_{ml}\,\hat{h}_{E_i}^{(m-k)}(l)\,e^{2i\pi\alpha_i l} \tag{3.10}$$

the $\hat{h}_{E_i}^{(n)}(m)$ are uniformly bounded. Thus, all Fourier coefficients converge and since only a finite number of them is nonzero, their sum exists and gives the desired $\tilde{h}_E^{(n)}(\theta)$.

On the other hand, if $n=q$, the nth Fourier component $\hat{h}_{E_i}^{(n)}(n)$ will in general diverge as $E_i \to E$. In fact, a straightforward computation shows that

$$(1-e^{-2\pi i\alpha q})\,\hat{h}_E^{(q)}(q)=q!\left(\frac{i}{4}\right)^q \sum_{l=1}^{q}\sum_{k_1=2}^{q}\cdots\sum_{k_l=2}^{q-k_1-\cdots-k_{l-1}} \times\binom{q}{k_1,\dots,k_l}\prod_{j=1}^{l-1}\frac{1}{1-e^{-2\pi i\alpha(k_1+\cdots+k_j)}}\prod_{i=1}^{l}\mathbf{E}(V^{k_i}) \tag{3.11}$$

For $\alpha\to p/q$, $1-e^{-2\pi i\alpha q}\to 0$, while the right-hand side of (3.11) approaches in general some finite limit. Thus, $\hat{h}_E^{(q)}(q)$ cannot remain finite as $\alpha\to p/q$.

Thus, the $\tilde{h}_E^{(n)}(\theta)$ cannot be the correct solution for the special energies $E=2\cos(p/q)\,\pi$. In fact, not even all the finite $\tilde{h}_E^{(n)}(\theta)$ can be correct. For, since the equation for $h_E^{(q)}(\theta)$ involves only the $h_E^{(k)}(\theta)$ with $k\leqslant q-2$, if $h_E^{(k)}(\theta)=\tilde{h}_E^{(k)}(\theta)$ for *all* $k\leqslant q-2$, then $h_E^{(k)}(\theta)$ would necessarily diverge. Thus, if there is an asymptotic expansion for $h_{\lambda,E}(\theta)$ with finite coefficients to all orders at the special energies, then some of the coefficients $h_E^{(n)}(\theta)$ for $n\leqslant q-2$ will have to be discontinuous as functions of the energy. This is the source of the so-called "anomalies." In the next section we show that Eq. (3.4) indeed permits a unique finite solution at the special energies. Moreover, we show that the anomaly occurs exactly in order $q-2$, i.e., for $n<q-2$, $h_E^{(n)}(\theta)=\tilde{h}_E^{(n)}(\theta)$!

4. PERTURBATION EXPANSION NEAR THE SPECIAL ENERGIES

We show now that the problems of perturbation theory near the special energies $E_0=2\cos\pi(p/q)$ can be avoided if, instead of starting the expansion with Eq. (2.2), we use Eq. (2.11) instead. Proceeding otherwise as in Section 3, this yields

$$\binom{n}{2}\left(\frac{d^2}{d\lambda^2}B^q_{\lambda,E_0,\varepsilon}\right)_{\lambda=0}\varphi_{E_0,\varepsilon}^{(n-2)}(x)=-\sum_{k=3}^{n}\binom{n}{k}\left(\frac{d^k}{d\lambda^k}B^q_{\lambda,E_0,\varepsilon}\right)_{\lambda=0}\varphi_{E_0,\varepsilon}^{(n-k)}(x) \tag{4.1}$$

Now note that

$$\left(\frac{d^2}{d\lambda^2}B^q_{\lambda,E_0,\varepsilon}\right)_{\lambda=0}\equiv A_{0,E_0,\varepsilon}=\sum_{k=0}^{q-1}T^k_{E_0}\left(\frac{1}{2}\frac{d^2}{dx^2}-\varepsilon\frac{d}{dx}\right)T^{q-k}_{E_0} \tag{4.2}$$

For a function f to be in the domain of this operator, f as well as $T^k_{E_0}f$, for $k=0,\dots,q-1$, must be twice differentiable. Moreover, since $\varphi_{\lambda,E}(x)$ is the density of a probability distribution, we are only interested in solutions of (4.1) in $L^1(\mathbf{R}, dx)$. In fact, it will turn out to be more convenient to seek solutions in Hilbert spaces $\mathscr{H}_{E_0,\varepsilon} \subset L^1(\mathbf{R}, dx)$ defined as follows: Let

$$\varphi^{(0)}_{E_0,\varepsilon}(x) = N_0 \frac{1}{(1+x^4)^{1/2}} \exp\left(\sqrt{2}\, 2\varepsilon \operatorname{arctg} \frac{\sqrt{2}\,x}{1-x^2}\right) \quad \text{if} \quad E_0 = 0 \tag{4.3}$$

$$= N_{E_0} \frac{1}{x^2 - E_0 x + 1} \exp\left(K_{E_0}\varepsilon \frac{1}{(4-E_0^2)^{1/2}} \operatorname{arctg} \frac{2x - E_0}{(4-E_0^2)^{1/2}}\right) \quad \text{if} \quad E_0 \neq 0 \tag{4.4}$$

Here N_{E_0} are normalization constants such that

$$\int_{-\infty}^{\infty} \varphi^{(0)}_{E_0,\varepsilon}(x)\, dx = 1$$

K_{E_0} is some finite constant that can be computed. Then

$$\mathscr{H}_{E_0,\varepsilon} = L^2\left(\mathbf{R}, \frac{1}{\varphi^{(0)}_{E_0,\varepsilon}(x)}\, dx\right) \tag{4.5}$$

In the remainder of this section we will prove the following two theorems:

Theorem I. The system of equations (4.1) has a unique normalized set of solutions $\varphi^{(n)}_{E_0,\varepsilon}(x)$ in the domain of $A_{0,E_0,\varepsilon}$ in $\mathscr{H}_{E_0,\varepsilon}$.

Theorem II. Let $\tilde{\varphi}^{(n)}_{E_0}(x) = (J_\alpha^{-1} \tilde{h}^{(n)}_{E_0})(x)$, where $\tilde{h}^{(n)}_{E_0}(\nu)$ are defined in Section 3. Then for $n \leqslant q-3$,

$$\varphi^{(n)}_{E_0,0}(x) = \tilde{\varphi}^{(n)}_{E_0}(x)$$

We first prove Theorem I. We start with the following lemma:

Lemma 4.1. $A_{0,E_0,\varepsilon}$ are self-adjoint operators on the Hilbert spaces $\mathscr{H}_{E_0,\varepsilon}$. The spectra of $A_{0,E_0,\varepsilon}$ are discrete, zero is a simple eigenvalue, and the normalized eigenfunction corresponding to it is $\varphi^{(0)}_{E_0,\varepsilon}(x)$. Moreover, if $\varepsilon = 0$, then

$$\sigma(A_{0,E_0,0}) = \{-C_{E_0} m^2 \mid m \in \mathbf{Z}\} \tag{4.6}$$

where C_{E_0} are constants and the eigenfunctions are

$$P_m(x) = \frac{1}{(1+x^4)^{1/2}} e^{2imF(x)} \quad \text{if} \quad E_0 = 0 \tag{4.7}$$

where

$$F(x)=\int_0^x \frac{1}{(1+t^4)^{1/2}}\,dt$$

and

$$P_m^{E_0}(x)=\frac{1}{x^2-E_0x+1}\exp[2im\,\mathrm{arccot}(x-\cot\pi\alpha)] \tag{4.8}$$

Lemma 4.2. The right-hand side of Eq. (4.1) is orthogonal to $\varphi^{(0)}_{E_0,\varepsilon}(x)$ in $\mathscr{H}_{E_0,\varepsilon}$.

Assuming these two lemmas, the proof of Theorem I is now easy: Since the spectrum of $A_{0,E_0,\varepsilon}$ is discrete, its inverse exists and is bounded on the subspace orthogonal to its kernel, i.e., orthogonal to $\varphi^{(0)}_{E_0\leqslant\varepsilon}(x)$. By Lemma 4.2, the right-hand side of (4.1) is in this subspace. Thus, all equations with $n\geqslant 1$ can be solved there by inverting $A_{0,E_0,\varepsilon}$. Since, moreover, all functions orthogonal to $\varphi^{(0)}_{E_0,\varepsilon}$ have integral zero, the normalization condition forces us to choose the normalized solution of the $n=0$ equation. This proves Theorem I. ∎

It remains to prove the lemmas.

We show first that $A_{0,E_0,\varepsilon}$ can always be written in the form

$$A_{0,E_0,\varepsilon}=\frac{d}{dx}\left(p(x)\frac{d}{dx}+s(x)\right) \tag{4.9}$$

with $p(x)$, $s(x)$ polynomials. Consider first the case $E_0=0$. A simple computation shows that

$$T_0\frac{d}{dx}T_0=\frac{d}{dx}x^2 \tag{4.10}$$

This together with (4.2) shows that $A_{0,0,\varepsilon}$ indeed has the form (4.9) and, moreover,

$$p(x)=1/2(1+x^4) \tag{4.11}$$

$$s(x)=x^3-\varepsilon(1+x^2) \tag{4.12}$$

If $E_0=2\cos\pi(p/q)$, (4.10) is simply replaced by

$$T_{E_0}^{-1}\frac{d}{dx}T_{E_0}=\frac{d}{dx}x^2 \tag{4.13}$$

Moreover,

$$T_{E_0}^{-1} A_{0,E_0,\varepsilon} T_{E_0} = A_{0,E_0,\varepsilon} \tag{4.14}$$

Using (4.13), one again derives (4.9) easily. A corollary of a more general result to be proven later (Lemma 4.5) will imply furthermore that

$$p(x) = c_{E_0}(x^2 - E_0 x + 1)^2 \tag{4.15}$$

and

$$s(x) = \frac{1}{2}\frac{d}{dx}\, p(x) - \varepsilon d_{E_0}(x^2 - E_0 x + 1) \tag{4.16}$$

with some constants c_{E_0} and d_{E_0}.

A straightforward computation shows now that $A_{0,E_0,\varepsilon}$ are symmetric operators in $\mathscr{H}_{E_0,\varepsilon}$. Self-adjointness is most easily seen by mapping $A_{0,E_0,\varepsilon}$ to the corresponding operator on periodic functions on the circle via J_α. The result is then a consequence of Sturm–Liouville theory.[8]

The general solution of the équation $A_{0,E_0,\varepsilon} f = 0$ is

$$f(x) = \left[\exp\left(-\int_0^x \frac{s(t)}{p(t)}\,dt\right)\right]\left[\int_0^x dt\,\frac{C}{p(t)}\exp\left(+\int_0^t \frac{s(y)}{p(y)}\,dy\right) + B\right] \tag{4.17}$$

However, only the solution with $C = 0$ is in the domain of $A_{0,E_0,\varepsilon}$. A solution g with $C \neq 0$ will not be "periodic" at infinity, i.e., $T_{E_0} g$ will not be continuous. Thus, zero is a simple eigenvalue of $A_{0,E_0,\varepsilon}$. In the same way one obtains the discreteness of the spectrum of $A_{0,\varepsilon}$. For $\varepsilon = 0$ we have computed all eigenfunctions and eigenvalues of $A_{0,E_0,0}$. It is easy to check that $P_m^{E_0}(x)$ given by (4.7), (4.8) are eigenfunctions. Moreover, they form a complete set in $\mathscr{H}_{E_0,\varepsilon}$. Since we will not use these results later, we omit the proof here. It suffices to mention that the P_m are is one to one correspondence with the Fourier basis on the circle.

This completes the proof of Lemma 4.1. ▮

Proof of Lemma 4.2. We only have to compute expressions

$$\int_{-\infty}^{\infty} \varphi_{E_0,\varepsilon}^{(0)}(x)\left[\left(\frac{d^k}{d\lambda^k} B_{\lambda,E_0,\varepsilon}^q\right)_{\lambda=0} \varphi_{E_0,\varepsilon}^{(n-k)}(x)\right]\frac{dx}{\varphi_{E_0,\varepsilon}^{(0)}(x)}$$

$$= \int_{-\infty}^{\infty} \frac{d}{dx} D^{(k-1)} \varphi_{E_0,\varepsilon}^{(n-k)}(x) = 0 \tag{4.18}$$

where we used the fact that by means of (4.13) we can write $[(d^k/d\lambda^k)\, B_{\lambda,E_0,\varepsilon}^q]_{\lambda=0}$ as $(d/dx)\, D^{(k-1)}$ with $D^{(k-1)}$ a differential operator of degree $k-1$ and that all the $\varphi_{E_0,\varepsilon}^{(n-k)}(x)$ produced in the previous steps of the recursion are periodic at infinity. ∎

We finally turn to the proof of Theorem II. We put $\varepsilon=0$ and supress this index. Let

$$\Pi_m \equiv \frac{1}{q} \sum_{n=0}^{q-1} \omega_q^{km} T_{E_0}^k \tag{4.19}$$

where $\omega_q = e^{(2\pi i/q)p}$, be the projectors on the eigenspaces of T_{E_0}. In particular, Π_0 projects on the functions that are invariant under T_{E_0}. Note that the $\tilde{\varphi}_{E_0}^{(n)}(x)$ satisfy $\Pi_0 \tilde{\varphi}_{E_0}^{(n)}(x)=0$ if $n>0$ (see Lemma 3.1). Our first observation is:

Lemma 4.3. Assume that $\varphi_{E_0}^{(k)}(x)=\tilde{\varphi}_{E_0}^{(k)}(x)$ for all $k<n$. Then

$$T_{E_0}(\tilde{\varphi}_{E_0}^{(n)}(x)-\varphi_{E_0}^{(n)}(x))=\tilde{\varphi}_{E_0}^{(n)}(x)-\varphi_{E_0}^{(n)}(x) \tag{4.20}$$

Proof. Both $\tilde{\varphi}_{E_0}^{(n)}$ and $\varphi_{E_0}^{(n)}(x)$ satisfy Eq. (3.4). The terms on the right of (3.4) involve only $\varphi_{E_0}^{(k)}(x)$, $\tilde{\varphi}_{E_0}^{(k)}(x)$, resp., with $k \leqslant n-2$, and thus coincide by assumption. Thus, subtracting the equations from each other, we arrive at (4.20). ∎

Lemma 4.4. Under the same assumption as in the previous lemma,

$$\tilde{\varphi}_{E_0}^{(n)}(x)=\varphi_{E_0}^{(n)}(x)$$

provided

$$\Pi_0 \sum_{n=3}^{n+2} \binom{n+2}{k} \left(\frac{d^k}{d\lambda^k} B_{\lambda,E_0}^q\right)_{\lambda=0} \varphi_{E_0}^{(n+2-k)}(x)=0 \tag{4.21}$$

Proof. If (4.20) holds, (4.1) implies that

$$\Pi_0 A_{0,E_0} \varphi_{E_0}^{(n)}(x)=0$$

But since Π_0 commutes with A_{0,E_0} [see (4.14)], this shows that $\Pi_0 \varphi_{E_0}^{(n)}(x)$ is in the kernel of A_{0,E_0}, i.e.,

$$\Pi_0 \varphi_{E_0}^{(n)}(x)=c\varphi_{E_0}^{(0)}(x)=c\tilde{\varphi}_{E_0}^{(0)}(x)$$

where the last equality holds if $n>0$ by assumption. But Lemma 4.3 shows now that

$$\tilde{\varphi}_{E_0}^{(n)}(x)-\varphi_{E_0}^{(n)}(x)=\Pi_0(\tilde{\varphi}_{E_0}^{(n)}(x)-\varphi_{E_0}^{(0)}(x))= -c\tilde{\varphi}_{E_0}^{(0)}(x)$$

Since the normalization condition implies that

$$\int_{-\infty}^{\infty} \tilde{\varphi}_{E_0}^{(n)}(x)\,dx = \int_{-\infty}^{\infty} \varphi_{E_0}^{(n)}(x)\,dx = 0$$

$c = 0$, and the lemma is proven. ∎

After this lemma the theorem will be proven if we can establish (4.20) for $n \leqslant q-3$. The crucial observation that will give this result is the following:

Lemma 4.5. Let $D^{(m-1)}$ be a differential expression in d/dx of degree $m-1$ with analytic coefficients. If $D^{(m-1)}$ satisfies

$$x^2 T_{E_0}^{-1} D^{(m-1)} T_{E_0} = D^{(m-1)} \tag{4.22}$$

and if $m < q$, then there are constants c_k such that

$$D^{(m-1)} = r_{E_0}(x) \sum_{k=1}^{m} c_k \left(\frac{d}{dx} r_{E_0}(x)\right)^{k-1} \tag{4.23}$$

where

$$r_{E_0}(x) = x^2 - E_0 x + 1 \tag{4.24}$$

Proof. By assumption

$$D^{(m-1)} = \sum_{i=1}^{m} p_i(x) \frac{d^{i-1}}{dx^{i-1}}$$

with analytic $p_i(x)$. We show first that (4.22) implies that

$$x^{2m} T_{E_0}^{-1}\, p_m(x)\, T_{E_0} = p_m(x) \tag{4.25}$$

Namely, using (4.13),

$$\begin{aligned} x^2 T_{E_0}^{-1}\, p_m(x) \frac{d^{m-1}}{dx^{m-1}}\, T_{E_0} &= x^2 T_{E_0}^{-1}\, p_m(x)\, T_{E_0} \left(\frac{d}{dx} x^2\right)^{m-1} \\ &= x^{2m} T_{E_0}^{-1}\, p_m(x)\, T_{E_0} \frac{d^{m-1}}{dx^{m-1}} + O\left(\frac{d^{m-2}}{dx^{m-2}}\right) \end{aligned} \tag{4.26}$$

where $O(d^{m-2}/dx^{m-2})$ is a differential expression of degree $m-2$. Comparing coefficients of d^{m-1}/dx^{m-1} gives (4.25). Next we show that (4.25) determines $p_m(x)$ up to a multiplicative constant. Rewriting (4.25) as

$$p_m\left(E_0 - \frac{1}{x}\right) = \frac{1}{x^{2m}}\, p_m(x) \tag{4.27}$$

and denoting by x_+ and x_- the two roots of $r_{E_0}(x)$, i.e.,

$$x_\pm = e^{\pm i\pi(p/q)} \tag{4.28}$$

we obtain by putting $x = x_\pm$ in (4.27)

$$p_m(x_\pm) = \frac{1}{(x_\pm)^{2m}}\, p_m(x_\pm) \tag{4.29}$$

and thus $p_m(x_\pm) = 0$ unless $(x_\pm)^{2m} \equiv e^{\pm i\pi 2m(p/q)} = 1$! This latter condition is not satisfied if $m < q$; $x_\pm$ are thus roots of p_m. Therefore,

$$p_m^{(1)}(x) \equiv p_m(x)/r_{E_0}(x) \tag{4.30}$$

is an analytic function. Moreover, it satisfies

$$p_m^{(1)}\left(E_0 - \frac{1}{x}\right) = \frac{1}{x^{2m-2}}\, p_m^{(1)}(x) \tag{4.31}$$

so that by the same argument as before, $x_\pm$ are roots of $p_m^{(1)}$. Continuing this process, we finally find that

$$p_m^{(m)}(x) \equiv \frac{p_m(x)}{[r_{E_0}(x)]^m} \tag{4.32}$$

is analytic and satisfies

$$p_m^{(m)}(E - 1/x) = p_m^{(m)}(x) \tag{4.33}$$

This equation admits only a constant as analytic solution, so that

$$p_m(x) = c_m[r_{E_0}(x)]^m \tag{4.34}$$

Thus, we may write

$$D^{(m-1)} = c_m r_{E_0}(x)\left[\frac{d}{dx}\, r_{E_0}(x)\right]^{m-1} + D^{(m-2)} \tag{4.35}$$

where $D^{(m-2)}$ is a differential expression of degree $m-2$ with analytic coefficients. Moreover, $D^{(m-2)}$ satisfies again (4.22) {since $r[(d/dx)\, r]^{m-1}$ does!}. We may thus proceed inductively to get (4.23).

Before we continue with the proof of Theorem II, we want to derive the following corollary, which has already been used to prove Theorem I.

Corollary. If $E_0 \neq 0$,

$$A_{0,E_0,\varepsilon} = c_1 \frac{d}{dx}\, r_{E_0}(x) \frac{d}{dx}\, r_{E_0}(x) + \varepsilon d_1 \frac{d}{dx}\, r_{E_0}(x) \tag{4.36}$$

Proof. Let first $\varepsilon = 0$. By (4.2) and using (4.13),

$$A_{0,E_0,0} = \frac{1}{2} \sum_{k=0}^{q-1} \frac{d}{dx} \gamma_k(x) \frac{d}{dx} \gamma_k(x)$$

$$= \frac{1}{2} \frac{d}{dx} \left[\sum_{k=0}^{q-1} \gamma_k^2(x) \frac{d}{dx} + \sum_{k=1}^{q-1} \gamma_k(x) \gamma_k'(x) \right] \tag{4.37}$$

where $\gamma_k(x)$ are polynomials defined by

$$\gamma_k(x) = x^2 T_{E_0}^{-1} \gamma_{k-1}(x)\, T_E, \qquad \gamma_0(x) = 1$$

On the other hand, Lemma 4.5 implies that

$$A_{0,E_0,0} = C_1 \frac{d}{dx} \left[r_{E_0}^2(x) \frac{d}{dx} + r_{E_0}(x)\, r_{E_0}'(x) \right] + C_2 \frac{d}{dx} r_{E_0}(x) \tag{4.38}$$

Comparing coefficients shows that $C_2 = 0$. If $\varepsilon \neq 0$, the second term in (4.36) is obtained by the same reasoning. ▮

This corollary establishes our earlier claims concerning the explicit form of $A_{0,E_0,\varepsilon}$.

We now continue with the proof of the theorem. By Lemma 4.5 if $m < q$,

$$\frac{d}{dx} D^{(m-1)} \tilde{\varphi}_{E_0}^{(0)}(x) = 0 \tag{4.39}$$

since $\tilde{\varphi}_{E_0}^{(0)}(x) = c/r_{E_0}(x)$.

The final step in our proof is thus to show that Eq. (4.2) can be written as

$$\frac{d}{dx} D^{(n+1)} \varphi_{E_0}^{(0)}(x) = 0$$

We consider first the term with $k = n + 2$ in (4.21), i.e.,

$$\Pi_0 \left(\frac{d^{n+2}}{d\lambda^{n+2}} B_{\lambda,E_0}^q \right)_{\lambda=0} \varphi_{E_0}^{(0)}(x) \tag{4.40}$$

$[(d^{n+2}/d\lambda^{n+2})\, B_{\lambda,E_0}^q]_{\lambda=0}$ is a differential operator of degree $n + 2$, with analytic coefficients, and can be decomposed into a sum of terms $\Delta_k^{(n+1)}$ such that

$$T_{E_0}^{-1}\, \Delta_k^{(n+1)} T_{E_0} = \omega_q^k\, \Delta_k^{(n+1)}$$

The terms with $k \neq 0$ do not contribute to (4.40), since $\tilde{\varphi}_{E_0}^{(0)}(x)$ is invariant under T_{E_0} and $\Delta_k^{(n+1)}\tilde{\varphi}_{E_0}^{(0)}(x)$ is therefore annihilated by Π_0. Using (4.13), we can write $\Delta_0^{(n+1)}$ as $(d/dx)\, D^{(n+1)}$, where $D^{(n+1)}$ is as in Lemma 4.5. Thus, (4.40) is equal to zero, *provided* $n+2<q$. We still have to consider the terms with $k<n+2$ in (4.21). Here we can use Eq. (3.4) to express $\tilde{\varphi}_{E_0}^{(n+2-m)}(x)$ in terms of differential operators acting on $\tilde{\varphi}_{E_0}^{(0)}(x)$ and then proceed as before. Namely, for $l<n$,

$$\Pi_m(1-T_{E_0})\, \tilde{\varphi}_{E_0}^{(l)}(x) = (1-\omega_q^m)\, \tilde{\varphi}_{E_0}^{(l)}(x)$$
$$= \Pi_m \sum_{k=2}^{l} \binom{l}{k} \left(\frac{d^k}{d\lambda^k} B_{\lambda,E_0} \right)_{\lambda=0} \tilde{\varphi}_{E_0}^{(l-k)}(x)$$

and hence

$$\tilde{\varphi}_{E_0}^{(l)}(x) = \sum_{m=1}^{q-1} \frac{1}{1-\omega_q^m} \Pi_m \sum_{k=2}^{l} \binom{l}{k} \left(\frac{d^k}{d\lambda^k} B_{\lambda,E_0} \right)_{\lambda=0} \tilde{\varphi}_{E_0}^{(l-k)}(x) \tag{4.41}$$

Since on the right of (4.41) only $\tilde{\varphi}_{E_0}^{(r)}$ with $r<l-2$ appears, we can iterate this process a finite number of times until we have expressed $\tilde{\varphi}_{E_0}^{(l)}(x)$ as a sum of terms with differential operators acting on $\tilde{\varphi}_{E_0}^{(0)}(x)$ only. Moreover, it is obvious from the construction that the degree of these operators is always less than or equal to l. Inserting $\tilde{\varphi}_{E_0}^{(l)}(x)$ then into the terms in (4.21) with $k<n$ gives the desired expression, i.e., differential operators of degree $\leqslant n+2$ acting on $\tilde{\varphi}_{E_0}^{(0)}(x)$. These vanish for the same reasons as the $k=n+2$ term. This concludes the proof of Theorem II. ∎

Remark. We can actually prove that $A_{\lambda,E_0,\varepsilon} \to A_{0,E_0,\varepsilon}$ in $\mathscr{H}_{E_0,\varepsilon}$ as $\lambda \to 0$, in the sense of Kato's strong resolvent convergence in the generalized sense. If we assume that the characteristic function of the probability distribution μ has fractional polynomial decay in addition to finite moment of all orders, it can be extracted from the work of Campanino and Klein[2] that

$$\sigma(A_{\lambda,E_0,\varepsilon}) \subset \{0\} \cup \{z; \operatorname{Re} z \leqslant -c\}$$

for λ small enough, where $c>0$ depends only on μ. We can also show that $(A_{\lambda,E_0,\varepsilon}-z)^{-1} \to (A_{0,E_0,\varepsilon}-z)^{-1}$ strongly for $\operatorname{Re} z \geqslant c' > 0$. But we were not able to prove strong resolvent convergence in a neighborhood of 0. If we could do so, then we could use perturbation theory to prove that the expansion we get from (4.1) is actually asymptotic.

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